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Vacuum Metrics Without Symmetry†

IVOR ROBINSON and JOANNA R. ROBINSON

Southwest Center for Advanced Studies, Dallas, Texas

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Abstract

Vacuum metrics admitting null, geodetic, expanding, shear-free congruences are investigated. A new solution is derived which contains three holomorphic functions of a complex variable. It includes, as special cases, a number of well-known solutions such as those of Kerr and Taub-NUT. In general, however, it admits no Killing vectors.

1. *Introduction*

With respect to a vector field k^r which is null, geodesic and expanding,‡

$$k_r k^r = 0, \qquad k_{r;s} k^s = 0, \qquad k^r{}_{;r} \neq 0 \tag{1.1}$$

the field equations for empty space,

$$R_{ab} = 0 \tag{1.2}$$

divide naturally into three groups: the *main equations*

$$k_{[a} R_{b][c} k_{d]} = 0 \tag{1.3}$$

the *trivial equation*

$$R = 0 \tag{1.4}$$

and the remaining *subsidiary conditions*. If the main equations are satisfied then, as Sachs (1962, 1963) has shown, the trivial equation reduces to an identity, and the coordinates can be specialized in such a way that only three of them enter into the subsidiary conditions.§

In the case where the basic vector field is shear-free,

$$k_{(a;b)} k^{a;b} = \tfrac{1}{2}(k^a{}_{;a})^2 \tag{1.5}$$

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‡ Throughout the paper, the following conventions are used: indices range and sum over 1, 2, 3, 4; a comma followed by an index or indices denotes partial differentiation, a semicolon covariant differentiation; round index-brackets indicate symmetrization over the indices enclosed, square brackets anti-symmetrization.

§ A simple proof of Sachs' Lemma is given in Appendix A.

the general solution to the main equations can be written explicitly; but the subsidiary conditions have proved less tractable. In all solutions hitherto published, either the basic vector field is hypersurface-orthogonal (Robinson & Trautman, 1960), or the metric admits a Killing vector (Taub, 1951; Newman, *et al.*, 1963; Kerr, 1963; Kerr & Schild, 1964; Demianski & Newman, 1966; Carter, 1967; Robinson, *et al.*, 1969). In the latter event one is able to reduce the number of independent variables in the subsidiary equations. Here we achieve the same effect by imposing covariant conditions on the metric which do not imply any symmetry. For a certain class, also defined covariantly, we obtain the general solution. This metric, which contains three pairs of arbitrary functions, does not in general admit any Killing vector. By appropriate choice of the disposable functions, however, one can obtain metrics with Killing fields (including all those cited above, with the possible exception of Carter's). As a by-product of the investigation, we solve what might be described as a modified Kerr-Schild problem: given a vacuum metric g_{ab} and a null, geodesic, shear-free, expanding vector field k^a, in what circumstances does there exist an H such that

$$g'_{ab} := g_{ab} + 2Hk_a k_b \tag{1.6}$$

is also a solution of the empty space equations?

2. *Solution of the Main Equations*

Assuming that there exists a congruence of curves Γ which is null, geodesic and shear-free, with an affinely normalized tangent vector k^r, we can find coordinates†

$$x^k = (\zeta, \tilde{\zeta}, \sigma, \rho) \tag{2.1}$$

for which

$$ds^2 = 2P^2\, d\zeta\, d\tilde{\zeta} + 2d\Sigma(d\rho + Z\, d\zeta + \tilde{Z}\, d\tilde{\zeta} + S\, d\Sigma) \tag{2.2}$$

$$d\Sigma := a(b\, d\zeta + \tilde{b}\, d\tilde{\zeta} + d\sigma) = k_r\, dx^r \tag{2.3}$$

with $a\ (\neq 0), b, \tilde{b}$ independent of ρ. Assuming also that k_r diverges, specializing the coordinates further, and using the notation

$$df = f_1\, d\zeta + f_2\, d\tilde{\zeta} + f_3\, d\Sigma \tag{2.4}$$

for any function f of $\zeta, \tilde{\zeta}, \sigma$, we can write the integral of the main equations as

$$P^2 = \exp(2u)(\rho^2 + \Omega^2), \tag{2.5}$$

$$Z = \rho\Lambda - i(\Omega_1 + \Lambda\Omega), \qquad \tilde{Z} = \rho\tilde{\Lambda} + i(\Omega_2 + \tilde{\Lambda}\Omega) \tag{2.6}$$

$$S = \rho u_3 - \tfrac{1}{4}(K + \tilde{K}) + (\rho m + \Omega M)/(\rho^2 + \Omega^2) \tag{2.7}$$

† The tilde may be read as complex conjugation; but none of our analysis depends on this interpretation; only its application to the real world.

with

$$\Omega := \tfrac{1}{2} i a \exp(-2u)(\tilde{b}_1 - b_2), \tag{2.8}$$

$$\Lambda := a^{-1} a_1 - a b_3, \qquad \tilde{\Lambda} := a^{-1} a_2 - a \tilde{b}_3 \tag{2.9}$$

$$K := 2 \exp(-2u) L_2, \qquad \tilde{K} := 2 \exp(-2u) \tilde{L}_1 \tag{2.10}$$

$$L := \Lambda - u_1, \qquad \tilde{L} := \tilde{\Lambda} - u_2 \tag{2.11}$$

$$M := \tfrac{1}{2}(K + \tilde{K})\Omega + \tfrac{1}{2}\exp(-2u)[(\Omega_1 + \Lambda\Omega)_2 + \tilde{\Lambda}(\Omega_1 + \Lambda\Omega) + (\Omega_2 + \tilde{\Lambda}\Omega)_1 + \Lambda(\Omega_2 + \tilde{\Lambda}\Omega)] \tag{2.12}$$

where u and m, like $a, b, \tilde{b}$, are disposable functions of $\zeta, \tilde{\zeta}, \sigma$.

3. *Coordinate Transformations and Invariants*

The coordinates have been constructed out of the metric and the congruence Γ by a procedure which determines them up to the product of a trivial transformation

$$x^a \to [\tilde{\zeta}, \zeta, \sigma, \rho] \tag{3.1}$$

with

$$x^a \to [f(\zeta), \tilde{f}(\tilde{\zeta}), \sigma, \rho] \tag{3.2}$$

and

$$x^a \to [\zeta, \tilde{\zeta}, h(\zeta, \tilde{\zeta}, \sigma), \rho g(\zeta, \tilde{\zeta}, \sigma)] \tag{3.3}$$

$f', \tilde{f}', g$ and $h_{,3}$ being non-zero. Among the invariants of (3.2) and (3.3), we find the scalars

$$\rho^{-1}\Omega, \rho^{-2}K, \rho^{-2}\tilde{K}, \rho^{-3}m, \rho^{-3}M, (\rho\, e^u)^{-4} I, (\rho\, e^u)^{-4}\tilde{I} \tag{3.4}$$

and the 1-forms

$$\rho\, d\Sigma, dW, \rho^{-1} L_3\, d\zeta, \rho^{-1}\tilde{L}_3\, d\tilde{\zeta}, (\rho\, e^u)^{-2} J_2\, d\zeta, (\rho\, e^u)^{-2}\tilde{J}_1\, d\tilde{\zeta} \tag{3.5}$$

where

$$I := J_{22} + 2\tilde{L}J_2, \qquad \tilde{I} := \tilde{J}_{11} + 2L\tilde{J}_1 \tag{3.6}$$

$$J := L_1 + L^2, \qquad \tilde{J} := \tilde{L}_2 + \tilde{L}^2 \tag{3.7}$$

and

$$dW := W_a\, dx^a := L\, d\zeta + \tilde{L}\, d\tilde{\zeta} + d(u + \ln\rho) \tag{3.8}$$

The directions of $d\zeta$ and $d\tilde{\zeta}$ are also invariant under (3.2) and (3.3). Consequently, using the notation

$$df = f_{/1}\, d\zeta + f_{/2}\, d\tilde{\zeta} + f_{/3}\, \rho\, d\Sigma + f_{/4}\, dW \tag{3.9}$$

for any $f(\zeta, \tilde{\zeta}, \sigma, \rho)$, we see that if f is invariant, so are

$$f_{/1}\, d\zeta, \; f_{/2}\, d\tilde{\zeta}, \; f_{/3}, f_{/4} \tag{3.10}$$

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Thus, for example, we construct the invariant form

$$C_r\,dx^r := -i(\rho^{-3}M)_{/1}\,d\zeta + i(\rho^{-3}M)_{/2}\,d\tilde{\zeta} + \tfrac{1}{2}(\rho\,e^u)^{-4}(I+\tilde{I})\,\rho\,d\Sigma + 3\rho^{-3}m\,dW + d(\rho^{-3}m) \tag{3.11}$$

4. *Subsidiary Conditions*

The remaining field equations can be written as

$$C_r = 0 \tag{4.1}$$

To exhibit their independence of ρ, we remark that

$$\rho^3\,C_r\,dx^r = A\,d\zeta + \tilde{A}\,d\tilde{\zeta} + B\,d\Sigma \tag{4.2}$$

where

$$A := (m-iM)_1 + 3\Lambda(m-iM), \qquad \tilde{A} := (m+iM)_2 + 3\tilde{\Lambda}(m+iM) \tag{4.3}$$

and

$$\begin{aligned} B &:= \exp(-3u)[\exp(3u)(m+iM)]_3 + \exp(-4u)\,I \\ &= \exp(-3u)\,[\exp(3u)(m-iM)]_3 + \exp(-4u)\,\tilde{I} \end{aligned} \tag{4.4}$$

We may express some of these results more conveniently by means of a function $U(\zeta,\tilde{\zeta},\sigma)$ such that

$$U_3 = \exp(-u) \tag{4.5}$$

The definitions (2.12), (3.6), (3.7) and (4.4) can now be replaced by

$$M = \tfrac{1}{2}i\exp(-3u)(U_{1122} - U_{2211}), \tag{4.6}$$

$$J = e^u\,U_{113}, \qquad \tilde{J} = e^u\,U_{223} \tag{4.7}$$

$$B\exp(3u) = [m\exp(3u) + \tfrac{1}{2}(U_{1122} + U_{2211})]_3 - \exp(-u)J\tilde{J} \tag{4.8}$$

In a given coordinate system, the metric determines the potential **U** up to the gauge transformation

$$U \to U - \gamma(\zeta,\tilde{\zeta}) \tag{4.9}$$

It follows directly from the definition (3.11) that

$$C_{[r,s]} + 3C_{[r}W_{s]} = 3\rho^{-3}m\,W_{[r,s]} + \text{terms independent of } m \tag{4.10}$$

Three components of this bivector are identically zero. Evaluating the others,† and using the subsidiary conditions, we obtain

$$\begin{aligned} 3\tilde{L}_3(m+iM) &= \exp(-4u)(I_2 + 4\tilde{L}I) \\ 3L_3(m-iM) &= \exp(-4u)(\tilde{I}_1 + 4L\tilde{I}) \end{aligned} \tag{4.11}$$

† By means of the commutation relations given in Appendix B.

and

$$\tfrac{3}{2}i(K-\tilde{K})\,m = \exp(-4u)(I+\tilde{I})\,\Omega - \exp(-2u)\{(M_1+3\Lambda M)_2 + 3\tilde{\Lambda}(M_1+3\Lambda M) + (M_2+3\tilde{\Lambda}M)_1 + 3\Lambda(M_2+3\tilde{\Lambda}M)\} \quad (4.12)$$

5. *Further Reduction of the Metric*

From now onwards, we shall consider the special case

$$L = L(\zeta, \tilde{\zeta}), \qquad \tilde{L} = \tilde{L}(\zeta, \tilde{\zeta}) \quad (5.1)$$

This is an invariant restriction, since it is equivalent to the vanishing of two of the forms listed in (3.5).

Under the transformations (3.2) and (3.3),

$$a \to a/gh_{,3}, \qquad u \to u - \tfrac{1}{2}\ln(f'\tilde{f}'g^2) \quad (5.2)$$

We may therefore specialize the coordinates ρ and σ so that

$$a = 1, \qquad u_3 = 0 \quad (5.3)$$

We now obtain

$$\begin{aligned} b &= -(L+u_{,1})\,\sigma + e^u\tilde{\beta}(\zeta,\tilde{\zeta}) \\ \tilde{b} &= -(\tilde{L}+u_{,2})\,\sigma + e^u\beta(\zeta,\tilde{\zeta}) \end{aligned} \quad (5.4)$$

from the definitions (2.9) and (2.11), and

$$\begin{aligned} m + iM &= -\exp(-4u)\,I\sigma + \exp(-3u)\,\alpha(\zeta,\tilde{\zeta}) \\ m - iM &= -\exp(-4u)\,\tilde{I}\sigma + \exp(-3u)\,\tilde{\alpha}(\zeta,\tilde{\zeta}) \end{aligned} \quad (5.5)$$

from the subsidiary condition $B = 0$ together with (4.4).

The metric is thus determined by seven functions of ζ and $\tilde{\zeta}$: its dependence on σ has been made explicit.

The integrability conditions (4.11), together with the definitions (3.6) and (3.7), now form a system which involves only the *background variables* L and $\tilde{L}$:

$$\begin{aligned} J &= L_{,1} + L^2, \qquad J = \tilde{L}_{,2} + \tilde{L}^2 \\ I &= J_{,22} + 2\tilde{L}J_{,2}, \qquad \tilde{I} = \tilde{J}_{,11} + 2L\tilde{J}_{,1} \end{aligned} \quad (5.6)$$

$$I_{,2} + 4\tilde{L}I = 0, \qquad \tilde{I}_{,1} + 4L\tilde{I} = 0 \quad (5.7)$$

When the *background equations* (5.7) hold, the subsidiary conditions $A = \tilde{A} = 0$ reduce to

$$\alpha_{,2} + 3\tilde{L}\alpha + I\beta = 0, \qquad \tilde{\alpha}_{,1} + 3L\tilde{\alpha} + \tilde{I}\tilde{\beta} = 0 \quad (5.8)$$

From (5.1), (5.3) and (5.4) we can compute M, most conveniently by using (4.5) and (4.6). On substituting into (5.5), we obtain the *compatibility condition*

$$(\beta_{,2}+\tilde{L}\beta)_{,11}-J(\beta_{,2}+\tilde{L}\beta)-2J_{,2}\beta+\alpha$$
$$=(\tilde{\beta}_{,1}+L\tilde{\beta})_{,22}-\tilde{J}(\tilde{\beta}_{,1}+L\tilde{\beta})-2\tilde{J}_{,1}\tilde{\beta}+\tilde{\alpha} \qquad (5.9)$$

This completes our system of field equations. We shall solve the *mass-aspect equations* (5.8) explicitly and reduce the compatibility condition to an equation in one dependent variable. The procedure depends on the background; but the reduced compatibility equation is the same throughout.

If we are dealing with a background for which

$$I\neq 0\neq \tilde{I} \qquad (5.10)$$

then we introduce variables ϕ and ψ in place of α and $\tilde{\alpha}$:

$$\alpha=-I(\phi+i\psi), \qquad \tilde{\alpha}=-\tilde{I}(\phi-i\psi) \qquad (5.11)$$

Using the background equations, we get

$$\beta=(\phi+i\psi)_{,2}-\tilde{L}(\phi+i\psi)$$
$$\tilde{\beta}=(\phi-i\psi)_{,1}-L(\phi-i\psi) \qquad (5.12)$$

for the mass-aspect equations, and

$$\psi_{,1122}-(\tilde{J}\psi)_{,11}-(J\psi)_{,22}+J\tilde{J}\psi=0 \qquad (5.13)$$

for the compatibility condition.

In the limiting case,

$$I=0=\tilde{I} \qquad (5.14)$$

the mass-aspect equations decouple. Assuming that

$$J_{,2}\neq 0\neq \tilde{J}_{,1} \qquad (5.15)$$

we may write their general solution as

$$\alpha=2(J_{,2})^{3/2}\nu(\zeta), \qquad \tilde{\alpha}=2(\tilde{J}_{,1})^{3/2}\nu(\tilde{\zeta}) \qquad (5.16)$$

We find it convenient to replace the variables β and $\tilde{\beta}$ by ϕ and ψ, writing

$$\beta=(J_{,2})^{1/2}\nu+(\phi+i\psi)_{,2}-\tilde{L}(\phi+i\psi)$$
$$\tilde{\beta}=(\tilde{J}_{,1})^{1/2}\tilde{\nu}+(\phi-i\psi)_{,1}-L(\phi-i\psi) \qquad (5.17)$$

Once more, the compatibility condition reduces to (5.13). A simple example of a solution of this kind is:

$$L=\tilde{L}=\tfrac{3}{2}(\zeta+\tilde{\zeta})^{-1}$$
$$\psi=(\zeta+\tilde{\zeta})[p(\zeta+\tilde{\zeta})^{\frac{1}{2}\sqrt{13}}+q(\zeta+\tilde{\zeta})^{-\frac{1}{2}\sqrt{13}}] \qquad (5.18)$$

where p and q are constants.

6. *Solutions with a Flat Background*

We shall obtain an explicit solution for the case

$$J_{,2} = 0 = \tilde{J}_{,1} \tag{6.1}$$

We begin by specializing the coordinates ζ and $\tilde{\zeta}$. Under the transformation (3.2),

$$J\,d\zeta^2 \to (J+F)\,d\zeta^2, \qquad \tilde{J}\,d\tilde{\zeta}^2 \to (\tilde{J}+\tilde{F})\,d\tilde{\zeta}^2 \tag{6.2}$$

where

$$\begin{aligned} F &:= \tfrac{1}{2}f'''/f' - \tfrac{3}{4}(f''/f')^2 \\ \tilde{F} &:= \tfrac{1}{2}\tilde{f}'''/\tilde{f}' - \tfrac{3}{4}(\tilde{f}''/\tilde{f}')^2 \end{aligned} \tag{6.3}$$

In appropriate coordinates, therefore,

$$J = 0 = \tilde{J} \tag{6.4}$$

We are still free to make transformations for which $F = \tilde{F} = 0$. This restriction is equivalent to

$$f(\zeta) = (p\zeta + q)/(r\zeta + s), \qquad \tilde{f}(\tilde{\zeta}) = (\tilde{p}\tilde{\zeta} + \tilde{q})/(\tilde{r}\tilde{\zeta} + \tilde{q}) \tag{6.5}$$

with constant coefficients such that $ps - qr \neq 0 \neq \tilde{p}\tilde{s} - \tilde{q}\tilde{r}$. Under the bilinear transformation,

$$L\,d\zeta \to [L - r/(r\zeta + s)]\,d\zeta, \qquad \tilde{L}\,d\tilde{\zeta} \to [\tilde{L} - \tilde{r}/(\tilde{r}\tilde{\zeta} + \tilde{s})]\,d\tilde{\zeta} \tag{6.6}$$

Thus, without loss of generality, we require that

$$L \neq 0 \neq \tilde{L} \tag{6.7}$$

We may now write the general solution of the background condition (6.1) and the mass-aspect equations as:

$$\begin{aligned} L &= [\zeta + \tilde{\lambda}(\tilde{\zeta})]^{-1}, \qquad \tilde{L} = [\tilde{\zeta} + \lambda(\zeta)]^{-1} \\ \alpha &= 2\mu(\zeta)\tilde{L}^3, \qquad \tilde{\alpha} = 2\tilde{\mu}(\tilde{\zeta})L^3 \end{aligned} \tag{6.8}$$

At this point we could reduce the compatibility condition to the standard form (5.13) by a change of variables. It is preferable, however, to make use of the potential $\mathbf{U}$. From (5.5) and (6.8),

$$M = -\tfrac{1}{2}i\exp(-3u)[(\mu\tilde{L})_{,22} - (\tilde{\mu}L)_{,11}] \tag{6.9}$$

Hence, using (4.6), (4.7) and (6.4), we obtain

$$\begin{aligned} (U_{11} + \mu\tilde{L})_{,3} &= 0, \qquad (U_{22} + \tilde{\mu}L)_{,3} = 0 \\ (U_{11} + \mu\tilde{L})_{,22} &= (U_{22} + \tilde{\mu}L)_{,11} \end{aligned} \tag{6.10}$$

which are the conditions for there to exist a function $\gamma(\zeta, \tilde{\zeta})$ such that

$$U_{11} + \mu\tilde{L} = \gamma_{,11}, \qquad U_{22} + \tilde{\mu}L = \gamma_{,22} \tag{6.11}$$

After a gauge transformation (4.9), therefore,

$$U_{11} + \mu\bar{L} = 0 = U_{22} + \bar{\mu}L \tag{6.12}$$

Integrating (4.5), we have

$$U = \exp(-u)\,\sigma + \phi(\zeta, \bar{\zeta}) \tag{6.13}$$

We now replace β and $\tilde{\beta}$ by new variables, χ and $\tilde{\chi}$, writing

$$\beta = \bar{L}(\chi - \phi) + \phi_{,2}, \qquad \tilde{\beta} = L(\tilde{\chi} - \phi) + \phi_{,1} \tag{6.14}$$

Evaluating U_{11} and U_{22}, we find from (6.12) that

$$\chi_{,2} = \bar{\mu}L/\bar{L}, \qquad \tilde{\chi}_{,1} = \mu\bar{L}/L \tag{6.15}$$

Thus our general solution contains three disposable functions of ζ (λ, μ and a function of integration contained in χ) and three of $\bar{\zeta}$. Like the solutions discussed in the previous section, it also involves two disposable functions of ζ and $\bar{\zeta}$, namely u and ϕ. These will require further consideration.

7. *Coordinate Conditions*

The background variables are invariant under the general transformation (3.3). Because of the co-ordinate conditions (5.3), this transformation is now restricted to the product of a change of scale,

$$x^a \to [\zeta, \bar{\zeta}, \sigma/g(\zeta, \bar{\zeta}), \rho g(\zeta, \bar{\zeta})] \tag{7.1}$$

and change in the origin of σ,

$$x^a \to [\zeta, \bar{\zeta}, \sigma + e^u h(\zeta, \bar{\zeta}), \rho] \tag{7.2}$$

Under the scale change, α, $\tilde{\alpha}$, β, $\tilde{\beta}$ are invariant; only u changes. Under the change of origin,

$$\alpha \to \alpha - Ih, \qquad \tilde{\alpha} \to \tilde{\alpha} - \bar{I}h$$
$$\beta \to \beta + h_{,2} - \bar{L}h \qquad \tilde{\beta} \to \tilde{\beta} + h_{,1} - Lh \tag{7.3}$$

which corresponds to a change in the variable ϕ alone.

We can give u a certain intrinsic significance by requiring that the Gaussian curvature of a 2-space with the metric

$$2\exp(2u)\,d\zeta\,d\bar{\zeta} \tag{7.4}$$

should be $\frac{1}{2}(K + \bar{K})$: this is equivalent to

$$(L + u_1)_2 + (\bar{L} + u_2)_1 = 0 \tag{7.5}$$

We thereby subject changes of scale to the restriction

$$(\ln g)_{,12} = 0 \tag{7.6}$$

The restricted transformation of ρ may also be written as

$$\rho \to \exp \Upsilon(\zeta, \tilde{\zeta}, \sigma, \rho) \tag{7.7}$$

with

$$\Upsilon_{/12} + \Upsilon_{/21} = \Upsilon_{/3} = 0, \qquad \Upsilon_{/4} = 1 \tag{7.8}$$

The integrability condition for these equations,

$$[\rho^{-2}(K + \tilde{K})]_{/3} = 0 \tag{7.9}$$

is satisfied because of (5.1).

In some cases we can specify Υ more completely. Suppose, for example, that

$$K = \tilde{K} \tag{7.10}$$

This, together with (5.1), is necessary and sufficient for dW, as defined in (3.8), to be integrable. We then have a scalar W such that

$$W_{/1} = W_{/2} = W_{/3} = 0, \qquad W_{/4} = 1 \tag{7.11}$$

and, by taking

$$\Upsilon = W \tag{7.12}$$

we determine ρ up to a multiplication by a constant. In this scale, σ disappears from all the basic functions except m:

$$a = 1, \qquad b_3 = \tilde{b}_3 = u_3 = 0, \tag{7.13}$$

$$m_3 = -I \exp(-4u) = -\tilde{I} \exp(-4u) = \text{constant}$$

For a general background subject to (5,10), we can fix ρ by taking

$$\Upsilon = u - \tfrac{1}{8} \ln(I\tilde{I}) + \ln \rho, \tag{7.14}$$

which is equivalent to

$$\exp(8u) = I\tilde{I} \tag{7.15}$$

Similarly, for a background subject to (5.14) and (5.15), we write

$$\exp(4u) = J_2 \tilde{J}_1 \tag{7.16}$$

and in the special case (6.1),

$$K\tilde{K} = 1 \tag{7.17}$$

except, of course, when

$$K\tilde{K} = 0 \tag{7.18}$$

Our earlier normalization (7.12) is consistent with (7.15) and (7.17) but not necessarily (7.16).

Considering that

$$(\rho^{-1} \Omega)_{/3} = \tfrac{1}{4} i \rho^{-2}(K - \tilde{K}) \tag{7.19}$$

we can fix the origin of σ, except where the background satisfies (7.10), by writing

$$\rho\sigma = -4i\rho^{-1}\Omega/\rho^{-2}(K - \tilde{K}) \tag{7.20}$$

Alternatively, for backgrounds subject to (5.10), we may write

$$\rho\sigma = \tfrac{1}{2}\rho \exp(4u)[I^{-1}(m + iM) + \tilde{I}^{-1}(m - iM)] \tag{7.21}$$

If neither of these procedures is available, then we can find coordinates in which all the components of the metric are independent of σ; and there is in principle no way of fixing its origin. We may still be able to impose some restrictions on changes of origin: in the case,

$$\Omega = 0 \tag{7.22}$$

for example, we put

$$d\sigma = d\Sigma \tag{7.23}$$

thereby requiring that $h(\zeta,\tilde{\zeta})$ should be constant.

8. *Remarks*

We can now reformulate the question posed in the Introduction, as follows: what are the conditions on a, b, $\tilde{b}$, u for the vector C_r defined in (3.11) to be invariant under the substitution

$$m \to m + H(\rho^2 + \Omega^2)/\rho \tag{8.1}$$

for some H not identically zero? Writing

$$h := (1 - \rho^{-2}\Omega^2)\rho^{-2}H \tag{8.2}$$

we have

$$C_r\,dx^r \to C_r\,dx^r + \mathrm{d}h + 3h\,dW \tag{8.3}$$

A necessary and sufficient condition, therefore, is that dW should be integrable; and this is equivalent to requiring that the metric should satisfy (7.13) in some coordinate system. There are thus two possibilities: if (5.10) holds, the transformation (8.1) is equivalent to changing the origin of σ by a constant; if not, both metrics of (1.6) have $\partial/\partial\sigma$ as a Killing vector.

An example of the second kind is given in (5.18). We get another class of examples by imposing the restriction (7.10) on the solutions found in Section 6: these are metrics characterized, in appropriate coordinates, by (7.13) and

$$\lambda = \tfrac{1}{2}\kappa\zeta^{-1}, \qquad \tilde{\lambda} = \tfrac{1}{2}\kappa\tilde{\zeta}^{-1}, \qquad \kappa = 1, 0 \text{ or } -1 \tag{8.4}$$

which have been given elsewhere (Robinson, *et al.*, 1969). A number of metrics mentioned in the Introduction belong to this class.

Unless λ and $\tilde{\lambda}$ satisfy these conditions or

$$\lambda'\tilde{\lambda}' = 0 \tag{8.5}$$

the coordinates ρ and σ are determined by the metric and the congruence Γ. The functions

$$N(\zeta) := \mu^2(\lambda')^{-3}, \qquad \tilde{N}(\tilde{\zeta}) := \tilde{\mu}^2(\tilde{\lambda}')^{-3} \tag{8.6}$$

are similarly determined, since

$$N = -2(m + iM)^2 \tilde{K}^{-3}, \qquad \tilde{N} = -2(m - iM)^2 K^{-3} \tag{8.7}$$

Having μ and $\tilde{\mu}$ at our disposal, we can always arrange that N and $\tilde{N}$ should be solvable for ζ and $\tilde{\zeta}$ respectively. Unless

$$\mu\tilde{\mu} = 0 \tag{8.8}$$

however, Γ is fixed by its relation to the Weyl tensor: it is tangential to a coincident pair of principal null directions (Goldberg & Sachs, 1962). In general, therefore, the coordinates are determined by the metric alone. There are then no Killing vectors.

Appendix A

The main equations are necessary and sufficient conditions for the existence of a vector λ^a such that

$$R_{ab} = 2\lambda_{(a}k_{b)} \tag{A.1}$$

From this equation and (1.1), we obtain

$$k^a G_a{}^b{}_{;b} = \tfrac{1}{2}Rk^a{}_{;a} \tag{A.2}$$

and

$$k_{[a}R_{b]}{}^s{}_{;s}k_c = S_{abc;r}k^r + S_{abc}k^r{}_{;r} - 2S^r{}_{c[a}k_{b];r} \tag{A.3}$$

where

$$S_{abc} := k_{[a}R_{b]c} \tag{A.4}$$

We now use the Bianchi identities

$$G^{ab}{}_{;b} = 0 \tag{A.5}$$

Since k^a has non-vanishing divergence, the trivial equation follows directly from (A.2). Because of (A.3), if

$$k_{[a}R_{b]c} = 0 \tag{A.6}$$

at any point, then it holds on the geodesic tangential to k_a through this point.

The last equation is equivalent to

$$G_{ab} = \lambda k_a k_b \tag{A.7}$$

for some scalar λ: hence, if it holds in a four-dimensional region, we derive

$$G_a{}^b{}_{;b} = (\lambda_{,r}k^r + \lambda k^r{}_{;r})k_a \tag{A.8}$$

from which it follows that if the last component of the field equation is satisfied at any point, it is also satisfied along the appropriate geodesic.

Appendix B

The invariant derivatives defined in Section 3 satisfy the following commutation relations:

$$f_{/12} - f_{/21} = -\exp(2u)[2i\rho\Omega f_{/3} + \tfrac{1}{2}(K - \tilde{K})f_{/4}] \tag{B.1}$$

$$f_{/13} - f_{/31} = -\rho^{-1} L_3 f_{/4} \tag{B.2}$$

$$f_{/23} - f_{/32} = -\rho^{-1} \tilde{L}_3 f_{/4}$$

$$f_{/14} - f_{/41} = 0 \tag{B.3}$$

$$f_{/24} - f_{/42} = 0$$

$$f_{/34} - f_{/43} = -f_{/3} \tag{B.4}$$

We remark, incidentally, that (7.19) follows from the Jacobi identity

$$f_{/[[12]3]} + f_{/[[31]2]} + f_{/[[23]1]} = 0 \tag{B.5}$$

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